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A Note on Three Conjectures by Gonchar on Rational Approximation

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We prove a conjecture of A. A. Gonchar on local rational approximation, and give a partial proof of a second conjecture concerning uniform convergence.
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In [1], Gonchar has stated three conjectures concerning rational approximation of analytic functions. Here we prove the first conjecture (in a somewhat more general form). In addition, we partially prove the second and third conjectures. The basic results to be used have been established in [2–5].

Following the formulation in [1], we assume that

$$ff(z) = \sum_{n=0}^{\infty} f_n z^{-n}, \qquad |z| > R_{ff}\left(R_{ff} = \overline{\lim_{n \to \infty}} \, |f_n|^{1/n} < \infty \right), \tag{1}$$

and that f is the complete analytic function corresponding to the function element ff. For $n \in \mathbb{N}$, we define $v_n(ff) := \sup\{v(ff \cdot r); r \in \mathcal{R}_n\}$, where $v(g)$ is the multiplicity of the zero of g at $z = 0$, and $\mathcal{R}_n$ is the collection of all rational functions of degree at most n.

For any $n \in \mathbb{N}$, there exists essentially a unique function $\pi_n \in \mathcal{R}_n$ such that $v(ff - \pi_n) = v_n(ff)$. It is called the nth diagonal Padé approximant to the function element (1) (cf. [6, Sect. 73]). If the Padé table associated with (1) is normal (cf. [6, Sect. 74, Satz 3]), then we have $v_n(ff) = 2n + 1$ for all $n \in \mathbb{N}$, but this is a rather special property of (1). It is easy to see that, in general, we have

$$v_n(ff) > n \qquad \text{for} \quad n \in \mathbb{N}. \tag{2}$$

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From a theorem on the block-structure of Padé tables [6, Sect. 74, Satz 2], it follows that there exists an infinite sequence $N \subseteq \mathbb{N}$, depending on ff, with

$$v_n(ff) > 2n \qquad \text{for} \quad n \in N. \tag{3}$$

The next theorem determines the asymptotic behavior of $v_n(ff)$, as $n \to \infty$, under assumptions on (1) which are far less restrictive than normality of the Padé table associated with (1).

THEOREM 1. *If ff is an element of a (possibly multi-valued) non-rational analytic function f, and if all singularities of f are contained in a set $S \subseteq \mathbb{C}$ of (logarithmic) capacity zero, then*

$$\lim_{n \to \infty} \frac{1}{n} v_n(ff) = 2. \tag{4}$$

Remarks. (1) Theorem 1 is a generalization of Conjecture 1 in [1]. There it is only assumed that f has a finite number of singular points.

(2) If f is an algebraic function, then, of course, its set of singularities is finite. In this case, Theorem 1 has been proved in [7] as a functional analog of the well-known Thue–Siegel–Roth theorem of number theory.

(3) If there are no further restrictions of ff beyond (1), then Eq. (4) will, in general, not be true. This can immediately be seen by considering lacunary power series (cf. [1, (4)]).

(4) In [1], the connection of Eq. (4) with a classification of possible lacunae in the sequence of Hankel determinants associated with (1) is sketched.

(5) With $|\cdot| := a^{v(\cdot)}$, $a > 1$, we can define a metric $d(ff, gg) := |ff - gg|$ on the set of all function elements (1). With respect to this metric, the Padé approximant is the best rational approximant (cf. [1]). This phenomenon is the motivation for the use of the term "local approximation" in [1].

Proof of Theorem 1. For any $n \in \mathbb{N}$, there exist (not necessarily unique) polynomials P_n and Q_n of degree $\leqslant n$, with

$$Q_n(z) f(z) - P_n(z) = O(z^{-(n+1)}) \qquad \text{for} \quad z \to \infty, \tag{5}$$

and

$$\pi_n(z) = \frac{P_n(z)}{Q_n(z)} \tag{6}$$

(cf. [6, Sect. 73, Satz 1]). It follows rather immediately from (5) (cf. [2, Lemma 3.12]) that, for $R_1 > R_{ff}$,

$$\oint_{|\zeta| = R_1} \zeta^l Q_n(\zeta) f(\zeta)\, d\zeta = 0 \qquad \text{for} \quad l = 0, ..., n-1, \tag{7}$$

and

$$f(z) - \pi_n(z) = \frac{Q_n(z)^{-2}}{2\pi i} \oint_{|\zeta| = R_1} \frac{Q_n(\zeta)^2 f(\zeta)}{\zeta - z}\, d\zeta = \frac{I_n(z)}{Q_n(z)^2} \tag{8}$$

for $|z| > R_1$, where $I_n(z)$ is defined by the second equality in (8). According to [3, Theorem 2], under the assumptions made in Theorem 1, all zeros of the polynomials Q_n, $n \in \mathbb{N}$, up to a vanishing proportion, cluster on a compact set, which is characterized by minimal (logarithmic) capacity and the property that f is single valued and analytic in its complement. In particular, it follows from [3, Theorem 2] that

$$\lim_{n \to \infty} \frac{1}{n} \deg v(Q_n) = 1. \tag{9}$$

Let now $E_R := \{|z| \geqslant R\}$, $R > R_1$, and let j_n, $n \in \mathbb{N}$, be the number of zeros of $I_n(z)$ on E_R. By [4, Theorem 2(iii)], under the assumptions made in Theorem 1,

$$\lim_{n \to \infty} \frac{j_n}{n} = \sigma. \tag{10}$$

Since, by (8), $2 \deg r(Q_n) - j_n \leqslant v_n(ff) \leqslant 2 \deg r(Q_n)$, (9) and (10) immediately imply (4). Q.E.D.

We shall now discuss the last two conjectures of [1], which are concerned with uniform approximation of ff or f, respectively, on sets E_R, $R > R_{ff}$ (ff is analytic on E_R). By $\rho_n(ff)$, $n \in \mathbb{N}$, we denote the best approximation of (1) on E_R in $\mathscr{R}_n$, i.e.,

$$\rho_n(ff) := \inf\{\|ff - r_n\|; r_n \in \mathscr{R}_n\}, \tag{11}$$

where $\|\cdot\|$ is the sup norm on E_R. If (1) satisfies the assumptions of Theorem 1, then, as proved in [8] (cf. also [9]), there exists a unique compact set $K \subseteq \{|z| \leqslant R_{ff}\}$ fulfilling the following requirements: (i) f is single-valued in the complement of K,

$$\operatorname{cap}(K, E_R) = \inf_V \operatorname{cap}(V, E_R), \tag{12}$$

where $\operatorname{cap}(\cdot, E_R)$ denotes the condenser or Green's capacity (cf. [10]), and the infimum in (12) is taken over all compact sets $V \subseteq \{|z| \leqslant R_{ff}\}$ satisfying (i) (with V replacing K) and, finally, (iii) K is contained in all compact sets V which satisfy (i) and (ii). Using generalized (multi-point) Padé approximants in [5, Theorem 2] we obtain

THEOREM 2. *Under the assumptions of Theorem* 1, *we have*

$$\overline{\lim_{n \to \infty}} \ \rho_n(ff)^{1/n} \leqslant a_{ff}^2 \tag{13}$$

with $a_{ff} := \exp(-\operatorname{cap}(K, E_R)^{-1})$.

Remarks. (1) Theorem 2 proves Conjecture 2 of [1] only for function elements (1) satisfying the assumptions of Theorem 1; for the other cases the conjecture remains open. We note that in [1], instead of (13) it has only been conjectured that

$$\underline{\lim}_{n \to \infty} \ \rho_n(ff)^{1/n} \leqslant a_{ff}^2. \tag{14}$$

Indeed, it may well be true that (14), but not (13), can be proved for any element

(1) The results of [11] indicate that the technique of the Padé approximation is not likely to supply a general proof for (13).

(2) The special definition of E_R is not really necessary in Theorem 2. Without essential complications the theorem can be extended to any continuum in the closed complex plane on which (1) has an analytic continuation.

Finally we comment on Conjecture 3 of [1]: One half of this conjecture, i.e., the assertion that $\overline{\lim}$ of the left side of [1, (7)] is $\leqslant a_{ff}^2$ has been proved by Theorem 2. It is possible to also prove the other half for the function elements of (1) that satisfy the assumptions of Theorem 1; but the proof is too long to be presented here, and will be published elswhere.

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